Self-similar variational perturbation theory for critical exponents

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We extend field theoretic variational perturbation theory by self-similar approximation theory, which greatly accelerates convergence. This is illustrated by recalculating the critical exponents of $O(N)$-symmetric ϕ^4 theory. From only three-loop perturbation expansions in $4-\epsilon$ dimensions, we obtain *analytic results* for the exponents, which are close to those derived recently from ordinary field-theoretic variational perturbational theory to seventh order. In particular, the specific-heat exponent is found to be in good agreement with best-measured exponent $\alpha \approx -0.0127$ of the specific-heat peak in superfluid helium, found in a satellite experiment. In addition, our analytic expressions reproduce also the exactly known large-N behavior of the exponents.

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I. INTRODUCTION

The precise calculation of critical exponents of phase transitions is an important theoretical task. On the one hand, these exponents provide us with basic information on the behavior of thermodynamic quantities in the vicinity of critical points. On the other hand, such calculations require the development of new mathematical techniques to master the resummation problem of divergent perturbation expansions (see, e.g., [1,2]). The comparison of the calculated exponents with experiment serves as a test for the validity and accuracy of the mathematical methods.

Recently, a powerful method has been developed by one of the authors called *field-theoretic variational perturbation theory* [3–5], which converts divergent weak-coupling into convergent strong-coupling expansions. This method presents a more powerful alternative to the previously used method of Padé-Borel resummation. The improvement comes from an efficient use of the knowledge on the approach of the strong-coupling limit, characterized by the Wegner exponents ω. Higher accuracy has been amply demonstrated by calculating the critical exponents [1,4,6], in particular by predicting the most accurately known exponent $\alpha \approx -0.0127$ [5] of the specific-heat peak in superfluid helium, found in a satellite experiment with a temperature resolution of nanoKelvin.

An important feature of any resummation method is the convergence of the renormalized sequence of approximants. This can be studied by considering the analytic properties of the sought function with respect to coupling and by involving the corresponding dispersion relations [1,7]. The field-theoretic variational perturbation theory was shown to possess an exponentially fast convergence (see the detailed proof in [8]).

In an independent development, the other of the authors has set up a general resummation scheme called *self-similar approximation theory* [9–13]. This method also exhibits a fast convergence, which has been demonstrated for a variety of problems in quantum mechanics, statistical physics, and mathematical finance (see the review-type papers [14,15]). The aim of the present joint paper is to combine the two approaches. The combination is expected to have the fastest convergence available so far.

In Sec. II, we give a brief reminder of the basic formulas of field-theoretic variational perturbation theory, which will be used as a basis for a further acceleration of the convergence via self-similar approximation theory to be reviewed in Sec. III. In Sec. IV, we develop the combination of the two methods, which is then applied in Sec. V to calculate the critical exponents of the $O(N)$-symmetric ϕ^4 field theory.

II. FROM WEAK TO STRONG COUPLING

Physical quantities of interest are usually derived from theories as divergent series in powers of some bare coupling constant g_B. These provide us with reliable results only for very small g_B. Critical phenomena, however, take place at infinitely large g_B in comparison with the mass, the inverse length scale of the fluctuations [1]. In order to overcome this difficulty, one has to reorganize the divergent weak-coupling series into a convergent strong-coupling expansion. Such a reorganization is provided by the field-theoretic variational perturbation theory [3–5,8], briefly summarized in this section to recall the principal formulas needed in what follows.

Consider a real function $f(g_B)$ of a real g_B, whose limit $f(\infty)$ we want to find from a divergent weak-coupling expansion up to order L,

$$f^{(L)}(g_B) = \sum_{n=0}^{L} f_n g_B^n \quad (g_B \to 0), \tag{1}$$

with $L=1,2,3,\ldots$ enumerating the maximally available order. Our aim is to find the behavior of $f(g_B)$ at $g_B \to \infty$. The expansion coefficients grow factorially with n so that the series (1) could make sense only for very small g_B.

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Field-theoretic variational perturbation theory is based on the introduction in Eq. (1) of a variational parameter K by the identical replacement

$$g_B \to \frac{g_B}{(K^2 + g_B r)^q}, \quad r \equiv \frac{1 - K^2}{g_B}, \tag{2}$$

where q is a parameter related to the critical Wegner exponents ω, which in the renormalization-group approach to critical phenomena governs the approach to scaling. The parameter q in this paper corresponds to $q/2$ in the original work [3]. After the replacement, the series (1) is reexpanded in powers of g_B at fixed r, and at the end r is again replaced by $(1-K^2)/g_B$. This procedure introduces an artificial dependence on the dummy parameter K which is fixed by searching for a plateau in K which becomes flatter and flatter for increasing order. The plateau is horizontal only for the correct choice of q, and this condition will determine the Wegner coefficient ω [16].

In the upcoming calculations, we shall work with a slightly different but completely equivalent replacement

$$g_B \to \frac{s}{(1 - g_B r)^q}, \quad r \equiv \frac{\sigma}{g_B}, \tag{3}$$

where σ is defined as a function of g_B,

$$\sigma \equiv \sigma(g_B) \equiv 1 - \left(\frac{s}{g_B}\right)^{1/q}. \tag{4}$$

Following the rules of field theoretic variational perturbation theory, we have to form the functions

$$F^{(L)}(g_B, s, q) \equiv f^{(L)}\left(\frac{s}{(1 - g_B r)^q}\right) \tag{5}$$

to be calculated with the prescription that the terms $g_B^n \equiv s^n/(1-g_B r)^{nq}$ in the truncated series (1) are reexpanded systematically in powers of g_B up to g_B^{L-n}. After this, we replace again $r \to \sigma/g_B$ and optimize the resulting function in the variational parameter s. Using the binomial expansion

$$(1 - \sigma)^p \simeq \sum_{m=0}^{L-n} C_m^p (-\sigma)^m,$$

$$C_m^p \equiv \frac{\Gamma(p+1)}{\Gamma(m+1)\Gamma(p-m+1)}, \tag{6}$$

we obtain explicitly

$$F^{(L)}(g_B, s, q) = \sum_{n=0}^{L} \sum_{m=0}^{L-n} C_m^{-nq} (-\sigma)^m f_n s^n. \tag{7}$$

This must be optimized in s, yielding an order-dependent function $s^{(L)}(g_B)$ and an associated $\sigma^{(L)}(g_B) \equiv 1 - [s^{(L)}(g_B)/g_B]^{1/q}$.

Note that Eqs. (2) and (3) are identities and not directly related to the functions appearing in the Symanzik-type transformations [17], in spite of a certain similarity.

Our aim is to find the behavior of Eq. (7) in the strong-coupling limit $g_B \to \infty$. From Eqs. (3) we observe that $\sigma \to 1$ as $g_B \to \infty$ since, as we shall see, the optimal s is finite. This allows us to calculate from Eq. (7) as a finite approximant $F^{(L)}(\infty, s, q)$. In what follows, we limit ourselves to the approximants of third order, since then all calculations can be done analytically. The first three approximants are explicitly

$$F^{(1)}(\infty, s, q) = f_0 + f_1 s, \tag{8}$$

$$F^{(2)}(\infty, s, q) = F^{(1)}(\infty, s, q) + q f_1 s + f_2 s^2, \tag{9}$$

$$F^{(3)}(\infty, s, q) = F^{(2)}(\infty, s, q) + \frac{1}{2} q(1+q) f_1 s + 2q f_2 s^2 + f_3 s^3. \tag{10}$$

The optimal values $s^{(L)} = s^{(L)}(\infty)$ are found either by extremization

$$\frac{\partial}{\partial s} F^{(L)}(\infty, s, q) \bigg|_{s = s^{(L)}} = 0, \tag{11}$$

or, when the latter has no real solutions, from the turning points

$$\frac{\partial^2}{\partial s^2} F^{(L)}(\infty, s, q) \bigg|_{s = s^{(L)}} = 0. \tag{12}$$

To second order, there exists an extremum at

$$s^{(2)} = -(1+q)\frac{f_1}{2f_2}. \tag{13}$$

To third order, there can be two possibilities. There is an optimal extremum $s = s^{(3)}$ at one of the roots of the cubic equation

$$3 f_3 s^2 + 2(1+2q) f_2 s + \frac{1}{2}(1+q)(2+q) f_1 = 0 \tag{14}$$

and a turning point at

$$s^{(3)} = -(1+2q)\frac{f_2}{3f_3}. \tag{15}$$

Usually, conditions (11) and (12) yield optimal values of $s^{(L)}$ alternatively for odd and even orders L, respectively (see [1,3–6,8,18]), and this will be the case in the upcoming applications of this paper.

After determining $s^{(L)}$, we obtain the optimized approximants

$$F^{(L)\mathrm{opt}}(\infty, q) \equiv F^{(L)}(\infty, s^{(L)}, q). \tag{16}$$

To second order, this is

$$F^{(2)\mathrm{opt}}(\infty, q) = f_0 - (1+q)^2 \frac{f_1^2}{4 f_2}, \tag{17}$$

and to third order, with $s^{(3)}$ of Eq. (15) (see Ref. [4]),

$$F^{(3)\mathrm{opt}}(\infty,q) = f_0 - (1+q)(1+2q)(2+q)\frac{f_1 f_2}{6f_3} + (1+2q)^3\frac{2f_2^3}{27f_3^2}. \tag{18}$$

For each approximant (16), we must also specify the parameter q. If the Wegner exponent were known from other sources, we could use this. Otherwise we must determine it order by order, which yields an L-dependent result $q=q^{(L)}$, so that the final approximants will be

$$F^{(L)\mathrm{opt}} \equiv F^{(L)\mathrm{opt}}(\infty,q^{(L)}). \tag{19}$$

The determination of $q^{(L)}$ proceeds as follows. If we expect the function $f(g_B)$ to be finite in the strong-coupling limit, which is the case for the critical exponents, then the logarithmic derivative, to be referred to as a β function [1],

$$\beta(g_B) \equiv \frac{d\log f(g_B)}{d\log g_B}, \tag{20}$$

must tend to zero for $g_B\to\infty$: $\beta(\infty)=0$. From the expansion (1), it is straightforward to derive

$$\beta^{(L)}(g_B) = \sum_{n=0}^{L}\beta_n g_B^n. \tag{21}$$

Depending on whether f_0 is nonzero or zero, the coefficients β_n, up to third order, are given either by

$$\beta_0 = 0, \quad \beta_1 = \frac{f_1}{f_0}, \quad \beta_2 = 2\frac{f_2}{f_0} - \frac{f_1^2}{f_0^2}, \tag{22}$$

$$\beta_3 = \frac{f_1^3}{f_0^3} - 3\frac{f_1 f_2}{f_0^2} + 3\frac{f_3}{f_0} \quad (f_0 \neq 0), \tag{23}$$

or by equations

$$\beta_0 = 1, \quad \beta_1 = \frac{f_2}{f_1}, \quad \beta_2 = 2\frac{f_3}{f_1} - \frac{f_2^2}{f_1^2}, \tag{24}$$

$$\beta_3 = \frac{f_2^3}{f_1^3} - 3\frac{f_2 f_3}{f_1^2} + 3\frac{f_4}{f_1} \quad (f_0 = 0). \tag{25}$$

Now we treat the expansions $\beta^{(L)}(g_B)$ in the same way as before $f^{(L)}(g_B)$. We form the approximants $B^{(L)\mathrm{opt}}(\infty,q)$ similar to the way of deriving Eq. (16). This is set $B^{(L)\mathrm{opt}}(\infty,q)$ equal to zero to ensure $\beta(\infty)=0$ in each approximation. This determines the proper parameters $q=q^{(L)}$. For instance,

$$q^{(2)} = 2\sqrt{\frac{\beta_0\beta_2}{\beta_1^2}} - 1. \tag{26}$$

Note that the logarithmic derivative for determining $q^{(L)}$ can be formed from any function of g_B with a constant strong-coupling limit [19], i.e., from any critical exponent, not just from the function $f(g_B)$ we want to resum at the moment. Usually, the function $g_R(g_B)$ relating g_B to the renormalized coupling constant g_R is most convenient, since it is known to highest order.

III. SELF-SIMILAR APPROXIMATION THEORY

Self-similar approximation theory [9–13] is based on constructing a sequence of optimized approximants, which contain instead of a variational parameter a *trial function*. The general idea of deriving convergent sequences of optimized approximants with the help of trial control functions has been suggested in [20]. The first step in the optimization procedure is reminiscent of the Euler-Lagrange variational method. But while the latter is a single-step procedure [21], the optimized perturbation theory runs via a sequence of better and better approximants.

In the last section, we have shown how to calculate a sequence of trial functions $\{F^{(L)}(g_B,s,q)\}$ by field-theoretic variational perturbation theory [see Eq. (7)]. From these, s and q can be determined as functions of g_B by optimization. In self-similar approximation theory [9–13], the approximants of different order are considered as a flow on the manifold of approximants, in which order L of the approximation plays the role of a discretized pseudotime. In this interpretation, the sequence of approximations behaves like a dynamical system. The higher approximations will be obtained by improving the entire control functions $s^{(L)}(g_B)$, even if we are only interested in the strong-coupling value $f(\infty)$, for which the previous method required only an optimal parameter $s^{(L)}(\infty)$, Thus we have to perform the optimization procedure for all g_B *before* going to the limit $g_B\to\infty$. Instead of Eqs. (11) and (12), we have to solve the full extremality condition

$$\left.\frac{\partial}{\partial s}F^{(L)}(g_B,s,q)\right|_{s=s^{(L)}(g_B)} = 0 \tag{27}$$

and, if this has no real solution, the turning point condition

$$\left.\frac{\partial^2}{\partial s^2}F^{(L)}(g_B,s,q)\right|_{s=s^{(L)}(g_B)} = 0, \tag{28}$$

to find the lowest approximation for the trial functions $s^{(L)}=s^{(L)}(g_B)$. More explicitly, we could also record the parameter q at which the optimization is done in the arguments and write the solution as $s^{(L)}(g_B,q)$. But we shall refrain from doing so to avoid cluttering the notation. For the same reason we shall omit, for a while, the argument q in $F^{(L)}(\infty,s,q)$.

Starting from the trial functions $s^{(L)}(g_B)$, we construct an approximation following the general scheme developed in [9–15]. We define the reonomic function $g_B=g_B^{(L)}(\phi)$ by the reonomic constraint

$$F^{(1)}(g_B,s^{(L)}(g_B)) = \phi, \tag{29}$$

where $F^{(1)}$ is the lowest nontrivial function in the sequence $\{F^{(L)}\}$. Now further define an entire sequence of functions

$$y^{(L)}(\phi) \equiv f^{(L)}(g_B^{(L)}(\phi),s^{(L)}(g_B^{(L)}(\phi))), \tag{30}$$

with the initial term $y^{(1)}(\phi)=\phi$. The set of all functions $y^{(L)}(\phi)$ for $L=1,2,3,\ldots$ constitutes a space $\mathcal{Y}\subset y^{(L)}(\phi,q)$ called *approximation space*. The pseudotime evolution in this space forms a group of self-similarity transformations,

$$y^{(L+p)}(\phi) = y^{(L)}(y^{(p)}(\phi)). \tag{31}$$

The property of self-similarity (31) guarantees the existence of a fixed point $y^* = y^{(L)*}$ [14,15], which has the property

$$y^* = y^{(L)}(y^*). \tag{32}$$

More explicitly, the fixed point satisfies

$$y^{(L)*} \equiv F^{(L)*}(g_B^{(L)}) \equiv F^{(L)*}(g_B^{(L)}, s^{(L)}(g_B^{(L)})), \quad L \geqslant 1. \tag{33}$$

It defines the desired self-similar approximant

$$f^{(L)*}(g_B) \equiv F^{(L)*}(g_B(\phi), s^{(L)}(g_B(\phi))). \tag{34}$$

In order to find the fixed point, we define a pseudovelocity of the approximation sequence by the finite difference

$$v^{(L)}(\phi) \equiv F^{L+1}(g_B^{(L)}(\phi), s^{(L)}(\phi)) - f^{(L)}(g_B^{(L)}(\phi), s^{(L)}(\phi)). \tag{35}$$

If the $\{y^{(L)}\}$ with discrete $L = 0, 1, 2, \ldots$ were a flow function $\{y^{(t)}\}$ of a continuous time $t \geqslant 0$, it would follow a time evolution equation

$$\frac{\partial}{\partial t} y^{(t)}(\phi) = v^{(t)}(y^{(t)}(\phi)). \tag{36}$$

The integral form of the latter can be presented as the evolution integral

$$\int_{y^{(L)}}^{y^{(L)*}} \frac{d\phi}{v^{(L)}(\phi, q)} = \frac{1}{L}. \tag{37}$$

If the parameter q is unknown, it must be determined from a simultaneous treatment of the β function (20). In this case, we determine a sequence of optimal parameters $q = q^{(L)*}$, leading to the self-similar approximants

$$f^{(L)*}(g_B) \equiv F^{(L)*}(g_B, s^{(L)}(g_B), q^{(L)*}). \tag{38}$$

For the purpose of determining critical exponents, we are only interested in $f(g_B)$ at $g_B \to \infty$ and go to the limit of Eq. (38), yielding

$$f^{(L)*} \equiv \lim_{g_B \to \infty} f^{(L)*}(g_B). \tag{39}$$

The self-similar approximant (39) replaces the previous optimized approximant (19) of field-theoretic variational perturbation theory.

From the definition of the pseudovelocity (35), it follows that for the calculation of the L-order self-similar approximant $f^{(L)*}$, we need to know $L+1$ orders of the expansion in Eq. (1). If L is the last available order, we shall use as an $(L+1)$st approximation the average of the previous ones,

$$f^{(L+1)*} = \frac{1}{2}(f^{(L-1)*} + f^{(L)*}). \tag{40}$$

This approximation is expected to be reliable if the approximants tend to the limit $L \to \infty$ in an alternating fashion, once from above and once from below. This is not *a priori* ensured, but happens in many examples. In the series for the critical exponents to be treated here, this seems to be true. Thus Eq. (40) will be used to obtain the highest approximant. A more refined mathematical foundation for the usage of Eqs. (39) and (40) is given in Refs. [22,23].

IV. COMBINING SELF-SIMILAR AND VARIATIONAL THEORIES

Let us now be explicit and improve the convergence of the sequence $\{f^{(L)\mathrm{opt}}\}$ of variational perturbation theory derived in Eqs. (16)–(19) by self-similar approximation theory to obtain a new sequence $\{f^{(L)*}\}$. The improvement is most drastic at the initial stages of the procedure, when $L \leqslant 3$, so that we shall restrict ourselves to these low orders. An additional advantage is that all formulas up to third order can be derived analytically.

Recall that, in contrast to Sec. II, we do not consider from the beginning the limit of $g_B \to \infty$, but retain the full g_B dependence of the functions (7),

$$F^{(1)}(g_B, s, q) = f_0 + f_1 s, \tag{41}$$

$$F^{(2)}(g_B, s, q) = F^{(1)}(g_B, s, q) + q f_1 \sigma s + f_2 s^2, \tag{42}$$

$$F^{(3)}(g_B, s, q) = F^{(2)}(g_B, s, q) + \frac{1}{2} q(1+q) f_1 \sigma^2 s + 2 q f_2 \sigma s^2 + f_3 s^3, \tag{43}$$

which reduce to Eqs. (8)–(10) for $g_B \to \infty$ since then $\sigma \to 1$. For arbitrary g_B, we must optimize $F^{(L)}$ in s. Since σ depends on s via the relation (4), we may look for the extremum in the two s and σ while satisfying the condition

$$\frac{d\sigma}{ds} = \frac{\sigma - 1}{qs}. \tag{44}$$

If this is done with the function $F^{(L)}$, we obtain an optimal function $s^{(L)}(g_B)$. From this we calculate the approximant

$$F^{(L)\mathrm{opt}}(g_B, q) \equiv F^{(L)}(g_B, s^{(L)}(g_B), q). \tag{45}$$

To lowest order $L = 1$, an optimal function usually does not exist. In this case, we shall use the next higher existing $s^{(2)}(g_B)$ to define the lowest approximant. In principle we could, of course, form an entire off-diagonal matrix of variational functions,

$$F^{(L,L')}(g_B, q) \equiv F^{(L)}(g_B, s^{(L')}(g_B), q). \tag{46}$$

of which the functions (16) in Sec. II are only diagonal elements,

$$F^{(L)\mathrm{opt}}(\infty, q) = F^{(L,L)}(\infty, q). \tag{47}$$

The optimal function $s^{(2)}(g_B)$ is determined by the extremality condition (27), which amounts to the equation

$$(1+q) f_1 \sigma(g_B) + 2 f_2 s = 0. \tag{48}$$

From this, we obtain the variational expression

$$F^{(2)\mathrm{opt}}(g_B,q) = F^{(2,2)}(g_B,q) = f_0 + f_1 s^{(2)} + \frac{1-q}{1+q} f_2 s^{(2)2}. \tag{49}$$

In addition, we determine a lowest approximant with a first-order trial function $s^{(1)}(g_B) = s^{(2)}(g_B)$,

$$F^{(1)\mathrm{opt}}(g_B,q) = F^{(1,2)}(g_B,q) = f_0 + f_1 s^{(2)}. \tag{50}$$

The third-order trial function $s^{(3)}(g_B)$ is given by the extremum-point condition (27), which yields

$$\frac{1}{2}(1+q)(2+q) f_1 \sigma^2 + 2(1+2q) f_2 \sigma s^{(3)} + 3 f_3 s^{(3)} = 0. \tag{51}$$

If Eq. (51) has no real solution, we apply the turning point condition (28) and solve

$$(1+q)(2+q) f_1 \sigma(\sigma-1) + 2(1+2q) f_2 s^{(3)}(\sigma - 1 + q\sigma) + 6 f_3 q s^{(3)2} = 0. \tag{52}$$

We now construct the approximation cascade following the rules of Sec. III. Accordingly, we set $y^{(1)} = F^{(1)}$, $y^{(2)} = F^{(2)}$, and $y^{(L)} = F^{(L)}$ for $L \geq 2$. Therefore, the reonomic constraint (29) reads $f_0 + f_1 s = \phi$, which defines the reonomic function $g_B^{(1)}(\phi)$ through the relation

$$s^{(2)}(g_B(\phi)) = \frac{\phi - f_0}{f_1}. \tag{53}$$

The first-order pseudovelocity as defined in Eq. (35) is $v^{(1)} = F^{(2,1)} - F^{(1,1)}$, which reads explicitly

$$v^{(1)}(\phi,q) = A_1(\phi - f_0)^2, \tag{54}$$

with

$$A_1 \equiv \frac{(1-q) f_2}{(1+q) f_1^2}. \tag{55}$$

Using the evolution integral (37), we find from self-similar approximant (33) the expression

$$F^{(1)*}(g_B,q) = f_0 + \frac{F^{(2,2)}(g_B,q) - f_0}{1 - A_1[F^{(2,2)}(g_B,q) - f_0]}, \tag{56}$$

where $F^{(2,2)}$ of Eq. (45) is given by Eq. (49). In the strong-coupling limit $g_B \to \infty$, this reduces to

$$F^{(1)*}(\infty,q) = f_0 - \frac{(1+q)^2 f_1^2}{(5-q)^2 f_2}. \tag{57}$$

Let us now determine the value of the parameter q, following the procedure described at the end of Sec. II, but with the difference that now we construct the self-similar approximants for the expansion (21) of the β function, which will be denoted by $B^{(L)*}(g_B,q)$. These have the same form as $F^{(L)*}(g_B,q)$, except that the expansion coefficients f_n are replaced by β_n. The parameters $q^{(L)*}$ are defined by the boundary condition

$$B^{(L)*}(\infty, q^{(L)*}) = 0. \tag{58}$$

To first order, the result is

$$q^{(1)*} = \frac{\sqrt{\beta_0\beta_2(4\beta_1^2 + 5\beta_0\beta_2)} - \beta_1^2}{\beta_1^2 + \beta_0\beta_2}. \tag{59}$$

Substituting this into Eq. (56), we obtain $f^{(1)*}(g_B) = F^{(1)*}(g_B, q^{(1)*})$, in agreement with Eq. (34). The final result is given by Eq. (39), that is, by the value $f^{(1)*} = f^{(1)*}(\infty)$.

The second-order velocity (35) is $v^{(2)} = F^{(3,2)} - F^{(2,2)}$, where

$$F^{(3,2)}(g_B,q) = F^{(2,2)}(g_B,q) + \frac{(1+q) f_1 f_3 - 2q f_2^2}{(1+q) f_1} s^{(2)3}, \tag{60}$$

which is valid for any g_B, in particular in the limit $g_B \to \infty$, where

$$F^{(3,2)}(\infty,q) = f_0 - (1+q)^3 \frac{f_1^2}{4 f_2}\left(\frac{f_1 f_3}{2 f_2^2} - \frac{1-q}{1+q}\right). \tag{61}$$

The explicit form of the velocity $v^{(2)}$ is now

$$v^{(2)}(\phi,q) = A_2(\phi - f_0)^3, \tag{62}$$

where

$$A_2 \equiv \frac{(1+q) f_1 f_3 - 2q f_2^2}{(1+q) f_1^4}. \tag{63}$$

From the evolution integral (37), we find here

$$F^{(2)*}(g_B,q) = f_0 + \frac{F^{(2,2)}(g_B,q) - f_0}{\sqrt{1 - A_2[F^{(2,2)}(g_B,q) - f_0]^2}}. \tag{64}$$

The value $q^{(2)*}$ follows from the strong-coupling condition (58), which leads to the equation

$$(1+q^{(2)*})^4(\beta_0^2\beta_1\beta_3 - 2\beta_0^2\beta_2^2 + \beta_1^4) + 2(1+q^{(2)*})^3\beta_0^2\beta_2^2 - 16\beta_0^2\beta_2^2 = 0. \tag{65}$$

Inserting the appropriate solution $q^{(2)*}$ into Eq. (64), we obtain $f^{(2)*}(g_B) = F^{(2)*}(g_B, q^{(2)*})$. And the limiting value (39) is $f^{(2)*} = f^{(2)*}(\infty)$. If only three orders of the expansion (1) are available, then $f^{(3)*}$ is defined by Eq. (40), as explained at the end of Sec. III.

V. APPLICATION TO CRITICAL EXPONENTS

The above theory will now be applied to evaluate the divergent perturbation expansions of the critical exponents of the $O(N)$-symmetric ϕ^4 field in $4-\epsilon$ dimensions. The expansions are power series in the bare coupling parameter g_B/μ^ϵ, where μ is some mass parameter to make g_B/μ^ϵ dimensionless. They can be found up to six loops in the textbook [1]. In this field-theoretic context, the parameter q in the transformation (3) is directly related to the Wegner exponent ω [24], which characterizes the strong-coupling behavior of the renormalized coupling

$$g(g_B) \simeq g(\infty) - \mathrm{const} \times \frac{\mu^\omega}{g_B^{\omega/\epsilon}} \quad (g_B/\mu^\epsilon \to \infty). \tag{66}$$

The relation is

$$\omega = \epsilon/q. \tag{67}$$

In the sequel, we shall set $\mu=1$. The starting point is the expansion [1] for the renormalized coupling constant, which we shall limit to g_B^4 for simplicity,

$$g_R(g_B) \simeq g_B + c_2 g_B^2 + c_3 g_B^3 + c_4 g_B^4, \tag{68}$$

where $g_B \to 0$ and the coefficients are

$$c_0 = 0, \quad c_1 = 1, \quad c_2 = -\frac{N+8}{3\epsilon}, \quad c_3 = \frac{(N+8)^2}{9\epsilon^2} + \frac{3N+14}{6\epsilon},$$

$$c_4 = -\frac{(N+8)^3}{27\epsilon^3} - \frac{4(N+8)(3N+14)}{27\epsilon^2} - \frac{33N^2+922N+2960+24(27N+88)\zeta(3)}{648\epsilon}, \tag{69}$$

with $\zeta(z)$ being the Riemann zeta function. The logarithmic derivative of Eq. (68) yields the β function

$$\beta^{(3)}(g_B) = \frac{d\log g(g_B)}{d\log g_B} = 1 + \beta_1 g_B + \beta_2 g_B^2 + \beta_3 g_B^3, \tag{70}$$

with the coefficients

$$\beta_1 = c_2, \quad \beta_2 = 2c_3 - c_2^2, \quad \beta_3 = c_2^3 - 3c_2 c_3 + 3c_4. \tag{71}$$

In second-order variational perturbation theory, we have $q^{(2)}$ given by Eq. (26), which yields

$$q^{(2)} = 2\sqrt{1+p\epsilon} - 1, \tag{72}$$

where the notation

$$p \equiv \frac{3(3N+14)}{(N+8)^2} \tag{73}$$

is used. The first-order self-similar approximant $q^{(1)*}$ is defined in Eq. (59), resulting in

$$q^{(1)*} = \frac{\sqrt{(1+p\epsilon)(9+5p\epsilon)} - 1}{2+p\epsilon}. \tag{74}$$

The corresponding Wegner exponents are

$$\omega^{(2)} = \frac{\epsilon}{2\sqrt{1+p\epsilon} - 1} \tag{75}$$

in the variational perturbation theory, and

$$\omega^{(1)*} = \frac{(2+p\epsilon)\epsilon}{\sqrt{(1+p\epsilon)(9+5p\epsilon)} - 1} \tag{76}$$

in the self-similar approximation theory.

The second-order optimized approximant

$$G^{(2)\mathrm{opt}}(\infty, q) = \frac{3(1+q)^2\epsilon}{4(N+8)}, \tag{77}$$

corresponding to the renormalized coupling (68), with the parameter (72), becomes

$$g^{(2)\mathrm{opt}} = \frac{3(\epsilon + p\epsilon^2)}{N+8}, \tag{78}$$

according to Eq. (19). The first-order self-similar approximant

$$G^{(1)*}(\infty, q) = \frac{3(1+q)^2\epsilon}{(5-q^2)(N+8)}, \tag{79}$$

with $q^{(1)*}$ from Eq. (74), results in $g_1^* = G_1^*(\infty, q^{(1)*})$, as in Eq. (39). It turns out that, because of the equality

$$\frac{(1+q^{(1)*})^2}{5-(q^{(1)*})^2} = 1 + p\epsilon, \tag{80}$$

the values $g^{(1)*}$ and $g^{(2)\mathrm{opt}}$ coincide. However, $g^{(2)*}$, following from Eq. (64), is different from $g^{(3)\mathrm{opt}}$.

We now turn to the perturbation expansions of the critical exponents ν and γ. Other exponents need not be treated since they can be found from the above using well-known scaling relations (see, e.g., [1,25,26]). We begin with ν^{-1}, for which we use the expansion [1] up to g_B^3,

$$\nu^{-1(3)} = f_0 + f_1 g_B + f_2 g_B^2 + f_3 g_B^3, \tag{81}$$

with the coefficients

$$f_0 = 2, \quad f_1 = -\frac{N+2}{3}, \quad f_2 = \frac{N+2}{9}\left(\frac{N+8}{\epsilon} + \frac{5}{2}\right), \tag{82}$$

$$f_3 = -\frac{N+2}{108}\left[\frac{4(N+8)^2}{\epsilon^2} + \frac{2(19N+122)}{\epsilon} + 3(5N+37)\right]. \tag{83}$$

Following the above procedure, we get the optimized strong-coupling value

$$\nu^{-1(2)*} = 2 - (1+q)^2 \frac{(N+2)\epsilon}{2(2N+16+5\epsilon)}, \tag{84}$$

which, for $q^{(2)}$ from Eq. (72), gives the variational perturbation result $\nu^{-1(2)\mathrm{opt}} = \nu^{-1(2)\mathrm{opt}}(\infty, q^{(2)})$. Its self-similar improvement reads

$$\nu^{-1(1)*}(\infty, q) = 2 - \frac{2(1+q)^2(N+2)\epsilon}{(5-q^2)(2N+16+5\epsilon)}. \tag{85}$$

After inserting $q^{(1)*}$ from Eq. (74), we obtain

$$\nu^{-1(1)*} = 2 - \frac{2(N+2)(1+p\epsilon)\epsilon}{2(N+8)+5\epsilon}. \tag{86}$$

Again, it turns out that $\nu^{-1(1)*} = \nu^{-1(2)\mathrm{opt}}$, but $\nu^{-1(2)*} \neq \nu^{(3)\mathrm{opt}}$, with $\nu^{-1(2)*}$ defined by Eq. (49).

Finally, we resum the perturbation expansion for the critical exponent $\gamma = \nu(2-\eta)$, which reads in the form [1]

$$\gamma(g_B) \simeq f_0 + f_1 g_B + f_2 g_B^2 + f_3 g_B^3, \tag{87}$$

with the coefficients

$$f_0 = 1, \quad f_1 = \frac{N+2}{6}, \quad f_2 = -\frac{N+2}{36}\left[\frac{2(N+8)}{\epsilon} + 4 - N\right], \tag{88}$$

$$f_3 = \frac{N+2}{432}\left[\frac{8(N+8)^2}{\epsilon^2} + \frac{4(106+N-2N^2)}{\epsilon} + 194 + N(2N+17)\right]. \quad (89)$$

Here we find the optimized approximant

$$\gamma^{(2)\mathrm{opt}}(\infty, q) = 1 + \frac{(1+q)^2(N+2)\epsilon}{4[2(N+8)+(4-N)\epsilon]} \quad (90)$$

and the self-similar approximant

$$\gamma^{(1)*} = 1 + \frac{(N+2)(1+p\epsilon)\epsilon}{2(N+8)+(4-N)\epsilon}. \quad (91)$$

Again $\gamma^{(1)*}$ coincides with the variational perturbation result $\gamma^{(2)\mathrm{opt}} = \gamma^{(2)\mathrm{opt}}(\infty, q^{(2)})$, whereas $\gamma^{(2)*}$ does not equal $\gamma^{(3)\mathrm{opt}}$.

By construction, the self-similar approximants $f^{(L)*}$ obtained from the evolution integral (30) possess the same ϵ expansion, up to the given order L, as the optimized approximant $f^{(L)\mathrm{opt}}$ of variational perturbation theory. This is evident from expressions (56) and (64). For the variational perturbation results, on the other hand, it was shown in Refs. [4–6,19] that all expansions in powers of ϵ coincide with the expansions derived in the renormalization-group approach to critical phenomena [1]. As a consequence, also the presently derived self-similar approximants $f^{(L)*}$ possess the exact ϵ expansions. This can easily be verified by an explicit calculation.

It is interesting to compare the $1/N$ expansions with the self-similar approximants for larger N. These expansions for the critical exponents ω, ν, γ, and η are presented in Fig. 1, where they are compared with our self-similar approximants as well as with the results of the sixth-order variational perturbation theory [1], and with those of Padé-Borel resummations [27,28]. Our third-order results have the same accuracy as those of sixth or higher orders, obtained by other resummation techniques. In the limit $N \to \infty$, our exponents coincide with the known exact values

$$\alpha = \frac{D-4}{D-2}, \quad \beta = \frac{1}{2}, \quad \gamma = \frac{2}{D-2}, \quad \delta = \frac{D+2}{D-2}, \quad (92)$$

$$\nu = \frac{1}{D-2}, \quad \eta = 0, \quad \omega = 4-D, \quad (93)$$

where D is dimensionality.

Critical exponents for finite N have been calculated by Padé-Borel resummation methods based on six- and seven-loop expansions in $D=3$ dimensions [29–32] or in five-loop expansions in $D=4-\epsilon$ dimensions [33–36]. In Refs. [30–32,36], Borel-Leroy transformation has been used, combined with a conformal mapping. Different variants of the optimized perturbation theory [20] have been used [37–40]. Self-similar exponential approximants were given in [41]. Computer simulations, based on the Monte Carlo lattice studies, were presented in [42]. The available results have been reviewed in Refs. [1,43,44].

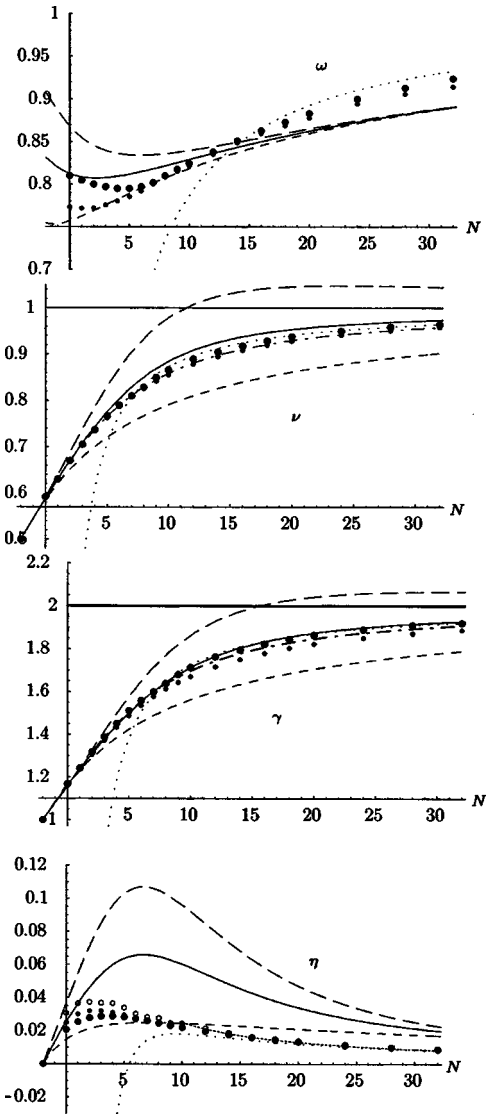

FIG. 1. Solid curves show our third-order approximations to $\omega, \nu, \gamma, \eta$. Short-dashed is second-, long-dashed curve is third-order approximation. Thin dots show sixth-order approximation of the textbook [1], fat dots the extrapolations to infinite order. The dash-dotted lines in the second and third figures are interpolations to the Padé-Borel resummations of [27,28] (where ω was not calculated). Their data for η scatter too much to be represented in this way—they are indicated by small circles in the fourth figure. The dotted curves show $1/N$ expansions of all four quantities. Note that our results lie closer to these than those of S.A. Antonenko and A.I. Sokolov. The solid η curve was calculated in the textbook [1] (see Fig. 20.2). The exact large-N limits are $\omega_{N=\infty}=4-D$, $\nu_{N=\infty}=1/(D-2)$, $\gamma_{N=\infty}=2/(D-2)$, and $\eta_{N=\infty}=0$. The exact values at $N=-2$ are $\nu_{N=-2}=1/2$, $\gamma_{N=-2}=1$, and $\eta_{N=-2}=0$ for all D.

A list of our results from the third-order self-similar improvements of variational perturbation theory for the critical exponents is given in Table I. The exponents ν, γ, and ω are calculated directly from their series, as is explained in the text. The other listed exponents are obtained from the scaling relations

TABLE I. Third-order critical exponents of self-similar variational perturbation results of this paper obtained from three-loop expansions in $4-\epsilon$ dimensions. Results are compared with the six-loop (for $N>3$) exponents and seven-loop exponents (for $N=0,1,2,3$) calculated in three dimensions in Refs. [3–5] and listed in the textbook [1]. We also show the exponents obtained by Padé-Borel resummation in Ref. [28], as well as earlier results (all cited in Notes and References). They refer to six-loop expansions in $D=3$ dimensions [29–32], or to five-loop expansions in $\epsilon=4-D$ [34,35]. The numbers in parentheses indicate the highest calculated approximation (seventh order for $N=0,1,2,3$ and sixth order for $N>3$) from which the final results were obtained by extrapolation to infinite order. The critical couplings g_c are different for calculations in $4-\epsilon$ and three dimensions due to different normalizations.

N	g_c	γ	η	ν	α	β	ω (ω_6)	
−2	0.758±0.037	1	0	1/2	$2-D/2$	$(D-2)/4$	0.831±0.077	
0	0.578±0.021	1.161±0.004	0.028±0.005	0.588±0.001	0.235±0.001	0.311±0.001	0.812±0.055	
		1.161(1.159)	0.0311±0.001	0.5886(0.5864)	0.234		0.810(0.773)	[5]
		1.168(1.159)	0.025(0.0206)	0.592(0.586)			0.810(0.7737)	[1]
	1.402	1.160	0.034	0.589	0.231	0.305		[27,28]
	1.421±0.004	1.161±0.003	0.026±0.026	0.588±0.001	0.236±0.004	0.302±0.004	0.794±0.06	[29]
	1.421±0.008	1.1615±0.002	0.027±0.004	0.5880±0.0015		0.3020±0.0015	0.80±0.04	[30–32]
		1.160±0.004	0.031±0.003	0.5885±0.0025		0.3025±0.0025	0.82±0.04	[36]
1	0.510±	1.238±0.004	0.037±0.009	0.630±0.008	0.109±0.012	0.327±0.004	0.808±0.046	
		1.241(1.236)	0.0347±0.001	0.6310(0.6270)	0.107		0.805(0.772)	[5]
		1.241(1.235)	0.030(0.0254)	0.630(0.627)			0.805(0.7724)	[1]
	1.419	1.239	0.038	0.631	0.107	0.327	0.781	[27,28]
	1.416±0.0015	1.241±0.004	0.031±0.011	0.630±0.002	0.110±0.008	0.324±0.06	0.788±0.003	[29]
	1.416±0.004	1.2410±0.0020	0.031±0.004	0.6300±0.0015		0.3250±0.0015	0.79±0.03	[30–32]
			0.035±0.002	0.628±0.001			0.80±0.02	[33–35]
		1.1239±0.004	0.037±0.003	0.6305±0.0025		0.3265±0.0025	0.81±0.04	[36]
2	0.454±0.012	1.310±0.019	0.045±0.012	0.671±0.018	−0.0124±0.0270	0.343±0.009	0.807±0.038	
		1.318(1.306)	0.0356±0.001	0.6713(0.6652)	−0.0129		0.800(0.772)	[5]
		1.318(1.306)	0.032(0.0278)	0.670(0.665)			0.800(0.7731)	[1]
	1.408	1.315	0.039	0.670	−0.010	0.348	0.780	[27,28]
	1.406±0.005	1.316±0.009	0.032±0.015	0.669±0.003	−0.007±0.009	0.346±0.009	0.78±0.01	[29]
	1.406±0.004	1.3160±0.0025	0.033±0.004	0.6690±0.0020		0.3455±0.002	0.78±0.025	[30–32]
			0.037±0.002	0.665±0.001			0.79±0.02	[33–35]
		1.315±0.007	0.040±0.003	0.671±0.005		0.3485±0.0035	0.80±0.04	[36]
3	0.407±0.010	1.378±0.037	0.052±0.015	0.709±0.030	−0.126±0.045	0.359±0.013	0.807±0.031	
		1.390(1.374)	0.0350±0.0005	0.7072(0.7004)	−0.122		0.797(0.776)	[5]
		1.387(1.372)	0.032(0.0288)	0.705(0.700)			0.797(0.7758)	[1]
	1.392	1.386	0.038	0.706	−0.117	0.366	0.780	[27,28]
	1.392±0.009	1.390±0.01	0.031±0.022	0.705±0.005	−0.115±0.015	0.362	0.78±0.02	[29]
	1.391±0.004	1.386±0.004	0.033±0.004	0.705±0.003		0.3645±0.0025	0.78±0.02	[30–32]
			0.037±0.002	0.79±0.02			0.79±0.02	[33–35]
		1.390±0.010	0.040±0.003	0.710±0.007		0.368±0.004	0.79±0.04	[36]
4	0.368±0.008	1.442±0.056	0.057±0.018	0.744±0.043	−0.232±0.064	0.374±0.018	0.809±0.026	
		1.451(1.433)	0.031(0.0289)	0.737(0.732)			0.795(0.780)	[1]
	1.375	1.449	0.036	0.738	−0.213	0.382	0.783	[27,28]
5	0.335±0.007	1.501±0.076	0.060±0.019	0.776±0.055	−0.328±0.082	0.388±0.022	0.812±0.022	
		1.511(1.487)	0.0295(0.0283)	0.767(0.760)			0.795(0.785)	[1]
	1.357	1.506	0.034	0.766	−0.297	0.396	0.788	[27,28]
6	0.306±0.006	1.554±0.095	0.062±0.020	0.804±0.066	−0.414±0.099	0.399±0.025	0.814±0.019 (0.792)	
		1.558(1.535)	0.0276(0.0273)	0.790(0.785)			0.797(0.792)	[1]
	1.339	1.556	0.031	0.790	−0.370	0.407	0.793	[27,28]
7	0.282±0.005	1.601±0.112	0.062±0.020	0.829±0.075	−0.489±0.0113	0.409±0.028	0.818±0.016	
		1.599(1.577)	0.0262(0.0260)	0.810(0.807)			0.802(0.800)	[1]

TABLE I. (*Continued.*)

N	g_c	γ	η	ν	α	β	ω (ω_6)	
	1.321	1.599	0.029	0.811	−0.434	0.417	0.800	[27,28]
8	0.261±0.004	1.643±0.127	0.061±0.019	0.851±0.083	−0.553±0.125	0.416±0.030	0.821±0.014	
		1.638(1.612)	0.0247(0.0246)	0.829(0.825)			0.810(0.808)	[1]
	1.305	1.637	0.027	0.830	−0.489	0.426	0.808	[27,28]
9	0.243±0.004	1.680±0.140	0.059±0.019	0.869±0.089	−0.608±0.134	0.422±0.032	0.825±0.012	
		1.680(1.643)	0.0233(0.0233)	0.850(0.841)			0.817(0.815)	[1]
	1.289	1.669	0.025	0.845	−0.536	0.433	0.815	[27,28]
10	0.226±0.003	1.713±0.150	0.057±0.017	0.884±0.093	−0.654±0.140	0.426±0.032	0.828±0.010	
		1.713(1.670)	0.0216(0.0220)	0.866(0.854)			0.824(0.822)	[1]
	1.275	1.697	0.024	0.859	−0.576	0.440	0.822	[27,28]
12	0.199±0.003	1.765±0.163	0.054±0.015	0.908±0.098	.0726±0.147	0.480±0.032	0.836±0.007	
		1.763(1.716)	0.0190(0.0198)	0.890(0.877)			0.838(0.835)	[1]
	1.249	1.743	0.021	0.881	−0.643	0.450	0.836	[27,28]
14	0.178±0.002	1.804±0.170	0.048±0.012	0.925±0.099	−0.777±0.148	0.486±0.031	0.843±0.006	
		1.795(1.750)	0.0169(0.0178)	0.905(0.894)			0.851(0.849)	[1]
	1.227	1.779	0.019	0.898	−0.693	0.457	0.849	[27,28]
16	0.160±0.002	1.833±0.172	0.042±0.010	0.938±0.097	−0.814±0.146	0.490±0.030	0.850±0.004	
		1.822(1.779)	0.0152(0.0161)	0.918(0.907)			0.862(0.860)	[1]
	1.208	1.807	0.017	0.911	−0.732	0.463	0.861	[27,28]
18	0.146±0.001	1.856±0.171	0.038±0.008	0.946±0.095	−0.840±0.142	0.492±−0.028	0.856±0.003	
		1.845(1.803)	0.0148(0.0137)	0.929(0.918)			0.873(0.869)	[1]
	1.191	1.829	0.015	0.921	−0.764	0.468	0.871	[27,28]
20	0.134±0.001	1.873±0.168	0.033±0.006	0.953±0.091	−0.861±0.137	0.493±0.026	0.862±0.002	
		1.864(1.822)	0.0125(0.0135)	0.938(0.927)			0.883(0.878)	[1]
	1.177	1.847	0.014	0.930	−0.789	0.471	0.880	[27,28]
24	0.114±0.001	1.898±0.158	0.027±0.004	0.963±0.084	−0.889±0.126	0.429±0.023	0.873±0.001	
		1.890(1.850)	0.0106(0.0116)	0.950(0.939)			0.900(0.894)	[1]
	1.154	1.874	0.012	0.942	−0.827	0.477	0.896	[27,28]
28	0.100±0.001	1.915±0.148	0.023±0.002	0.969±0.077	−0.907±0.116	0.427±0.021	0.882±0.000	
		1.909(1.871)	0.009232 (0.01010)	0.959(0.949)			0.913(0.906)	[1]
	1.136	1.893	0.010	0.951	−0.854	0.481	0.909	[27,28]
∞	0	$2/(D-2)$	0	$1/(D-2)$	$(D-4)/(D-2)$	1/2	$4-D$	

$$\alpha = 2 - \nu D, \quad \beta = \frac{\nu}{2}(D - 2 + \eta), \quad \gamma = \nu(2 - \eta). \quad (94)$$

The error bars are defined by the difference between $f^{(L)*}$ and $f^{(L-1)*}$. Our results are compared with those of the field-theoretic variational perturbation theory based on six-loop (for $N>3$) and seven-loop (for $N=0,1,2,3$) expansions calculated in three dimensions in Refs. [3–5] and listed in the book [1]. We also show the exponents recently obtained by Padé-Borel resummation [28], as well as earlier results [29–36], based on six-loop expansions in $D=3$ dimensions [29–32] and on five-loop expansions in $D=4-\epsilon$ dimensions [33–36]. The numbers in parentheses indicate the highest calculated approximation (seventh order for $n=0,1,2,3$ and sixth order for $N>3$), from which the effective extrapolations to infinite order were obtained as described in the book [1]. Comparing the results, we see that our third-order self-similar approximants yield the values for the critical exponents, which are close to those derived by other resummation techniques of sixth or seventh order. In the limiting cases of $N=-2$ and $N=\infty$, our results coincide with the known exact values of the critical exponents.

The critical exponents obtained by our method from three-loop perturbation expansions are all in good agreement with all experiments. Unfortunately, most of them are not sufficiently accurate to distinguish between different theoretical approaches. The most accurately known experiment is the measurement of the specific heat of liquid helium with nanoKelvin temperature resolution near the lambda point, which were performed in a satellite orbiting around the Earth [45,46]. The specific-heat exponent initially extracted from the data in [45] was $\alpha=-0.010\,56\pm0.0004$. This differed slightly from the result of seven-loop variational perturbation

theory obtained from three-dimensional ϕ^4 theory, which yielded $\alpha=-0.0129\pm0.0006$ [5]. However, a recently performed reanalysis of the data in Ref. [46] found $\alpha=-0.0127\pm0.0003$, thus confirming with great precision the theoretical result of Ref. [5]. Our present result $\alpha=-0.0124$ obtained in third-order self-similar-improved variational perturbation theory is again in perfect agreement with the latest experimental result. This is quite remarkable since the five-loop calculations in $4-\epsilon$ dimensions gave the value $\alpha=-0.013$ [41]. This illustrates the acceleration of the convergence of variational perturbation theory by the self-similar improvement developed in this paper. Note that calculating a small value of α is a rather complicated task, so that the error bars for α are usually quite large, which is also the case in our calculations, where the error bar is about 100% of α.

The main concern of this paper has been to demonstrate how the acceleration of the convergence can be achieved by combining two methods, each of which provides sufficiently fast convergence. For the N-component field theory, with not too large N, our third-order results are close to those of higher orders in other resummation techniques. Under the order, we mean the number of loops involved in the derivation of the series, employed in further resummation. Our results for ω and η at intermediate $N\sim5-50$ deviate slightly from those obtained in extrapolating the sixth-order approximation in the book [1]. However, these values of N are less physically interesting than the lower N values, where our results practically coincide with the sixth-order ones [1]. And, we would like to stress that in the limit $N\to\infty$, our results yield the known exact values of the exponents.

The accuracy of our calculations could be improved by employing the higher-loop expansions for the initial series. This, however, requires the usage of more complicated numerical calculations, which would be outside the scope of the present paper. Here we would like to emphasize that the acceleration of convergence can be achieved already at the very beginning of the resummation procedure, where the analytical treatment is still admissible.

In conclusion, we have developed a method for resumming divergent perturbation expansions. It combines field-theoretical variational perturbation theory with self-similar approximation theory, and accelerates greatly the convergence of either method by itself. The acceleration is especially useful if only low-order expansions are available due to the complexity of the problem. Up to third order, all results are found analytically. The method was illustrated by calculating the critical exponents whose third-order approximants are found to be close to the sixth- or seventh-order approximants of other resummation techniques. The specific-heat critical exponent α for $N=2$ is found to be in perfect agreement with the most accurately measured experimental value of α for superfluid helium.

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